

Hom-Lie superalgebra structures on exceptional simple Lie superalgebras of vector fields

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Abstract: In this paper, the Hom-Lie superalgebra structures on exceptional simple Lie superalgebras of vector fields are studied. Taking advantage of the \mathbb{Z} -grading structures and the transitivity, we prove that there is only the trivial Hom-Lie superalgebra structures on exceptional simple Lie superalgebras. Our proof is obtained by studying the Hom-Lie superalgebra structures on their 0-th and -1 -st \mathbb{Z} -components.

Keywords: exceptional Lie superalgebra, Hom-Lie superalgebra structure, gradation, transitivity.

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1. Introduction

The motivations to study Hom-Lie algebras and related algebraic structures come from physics and deformations of Lie algebras, in particular, Lie algebras of vector fields. Recently, this kind of investigation becomes rather popular [1–10], due in part to the prospect of having a general framework in which one can produce many types of natural deformations of Lie algebras, in particular, q -deformations. The notion of Hom-Lie algebras was introduced by J. T. Hartwig, D. Larsson and S. D. Silvestrov in [1] to describe the structures on certain deformations of Witt algebras and Virasoro algebras, which are applied widely in the theoretical physics [11–14]. Later, W. J. Xie and Q. Q. Jin gave a description of Hom-Lie algebra structures on semi-simple Lie algebras [9, 10]. In 2010, F. Ammar and A. Makhlouf generalized Hom-Lie algebras to Hom-Lie superalgebras [15]. In 2013 and 2015, B. T. Cao, L. Luo, J. X. Yuan and W. D. Liu studied the Hom-structures on finite dimensional simple Lie superalgebras [16, 17].

As is well known, Kac classified the infinite-dimensional simple linearly compact Lie superalgebras of vector fields, including eight Cartan series and five exceptional simple Lie

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superalgebras [18]. In 2014, the Hom-Lie superalgebra structures on the eight infinite-dimensional Cartan series were investigated by J. X. Yuan, L. P. Sun and W. D. Liu [19]. In the present paper, we characterize the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields. Taking advantage of the \mathbb{Z} -grading structures and the transitivity, we analyse the Hom-Lie superalgebra structures through computing the Hom-Lie superalgebra structures on the 0-th and -1 -st \mathbb{Z} -components. The main result of this study shows that the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields must be trivial.

2. Preliminaries

Throughout the paper, we will use the following notations. \mathbb{F} denotes an algebraically closed field of characteristic zero. $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of two-elements. The symbol $|x|$ and $\text{zd}(x)$ denote the \mathbb{Z}_2 -degree of a \mathbb{Z}_2 -homogeneous element x and the \mathbb{Z} -degree of a \mathbb{Z} -homogeneous element x , respectively. Notation $\langle v_1, \dots, v_k \rangle$ is used to represent the linear space spanned by v_1, \dots, v_k over \mathbb{F} . Now, let us review the five infinite-dimensional exceptional simple \mathbb{Z} -graded Lie superalgebras briefly. More detailed descriptions can be found in [18, 20].

We construct a Weisfeiler filtration of L by using open subspaces:

$$L = L_{-h} \supset L_{-h+1} \supset \dots \supset L_0 \supset L_1 \supset \dots$$

where L is a simple linearly compact Lie superalgebra. Let $\mathfrak{g}_i = L_i/L_{i+1}$, $\text{Gr}L = \bigoplus_{i=-h}^{\infty} \mathfrak{g}_i$. In [18], all possible choices for nonpositive part $\mathfrak{g}_{\leq 0} = \bigoplus_{i=-h}^0 \mathfrak{g}_i$ of the associated graded Lie superalgebra of L were derived. The five exceptional simple Lie superalgebras are as follows (the subalgebra $\mathfrak{g}_{\leq 0}$ is written below as the $h+1$ -tuple $(\mathfrak{g}_{-h}, \mathfrak{g}_{-h+1}, \dots, \mathfrak{g}_{-1}, \mathfrak{g}_0)$):

$$\begin{aligned} E(5, 10) : & \quad (\mathbb{F}^{5^*}, \wedge^2(\mathbb{F}^5), \mathfrak{sl}(5)) \\ E(3, 6) : & \quad (\mathbb{F}^{3^*}, \mathbb{F}^3 \otimes \mathbb{F}^2, \mathfrak{gl}(3) \oplus \mathfrak{sl}(2)) \\ E(3, 8) : & \quad (\mathbb{F}^2, \mathbb{F}^{3^*}, \mathbb{F}^3 \otimes \mathbb{F}^2, \mathfrak{gl}(3) \oplus \mathfrak{sl}(2)) \\ E(1, 6) : & \quad (\mathbb{F}, \mathbb{F}^n, \mathfrak{cso}(n), n \geq 1, n \neq 2) \\ E(4, 4) : & \quad (\mathbb{F}^{4|4}, \hat{P}(4)) \end{aligned}$$

As we know, the five exceptional simple Lie superalgebras are transitive and irreducible. In particular, $\mathfrak{g}_{-i} = \mathfrak{g}_{-1}^i$ for $i \geq 1$ and the transitivity will be used frequently in this paper:

$$\textbf{transitivity} : \text{ if } x \in \mathfrak{g}_i, i \geq 0, \text{ then } [x, \mathfrak{g}_{-1}] = 0 \text{ implies } x = 0. \quad (2.1)$$

To study the Hom-Lie superalgebras structure, we recall the following definition [16].

A Hom-Lie superalgebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \sigma)$ consisting of a \mathbb{Z}_2 -graded vector space \mathfrak{g} , an even bilinear mapping: $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and an even linear mapping: $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

$$\sigma[x, y] = [\sigma(x), \sigma(y)], \quad (2.2)$$

$$[x, y] = -(-1)^{|x||y|}[y, x], \quad (2.3)$$

$$(-1)^{|x||z|}[\sigma(x), [y, z]] + (-1)^{|y||x|}[\sigma(y), [z, x]] + (-1)^{|z||y|}[\sigma(z), [x, y]] = 0. \quad (2.4)$$

for all homogeneous elements $x, y, z \in \mathfrak{g}$.

An even linear mapping σ on a Lie superalgebra \mathfrak{g} is called a Hom-Lie superalgebra structure on \mathfrak{g} if $(\mathfrak{g}, [\cdot, \cdot], \sigma)$ is a Hom-Lie superalgebra. In particular, σ is called trivial if $\sigma = \text{id}|_{\mathfrak{g}}$.

It is obvious that σ is graded and if σ is a Hom-Lie superalgebra structure on a simple Lie superalgebra \mathfrak{g} , then σ must be a monomorphism.

Lemma 2.1. *Let $\mathfrak{g} = \oplus_{i \geq -h} \mathfrak{g}_i$ be a transitive Lie superalgebra. If σ is a Hom-Lie superalgebra structure on \mathfrak{g} and $\sigma|_{\mathfrak{g}_{-1}} = \text{id}|_{\mathfrak{g}_{-1}}$, then $\sigma(x) - x \in \mathfrak{g}_{-k}$, where $x \in \mathfrak{g}_0$, $k \geq 1$.*

Proof. For any $y \in \mathfrak{g}_{-1}$, from $\sigma|_{\mathfrak{g}_{-1}} = \text{id}|_{\mathfrak{g}_{-1}}$ and (2.2), one can deduce

$$[\sigma(x), y] = [\sigma(x), \sigma(y)] = \sigma[x, y] = [x, y].$$

Then $[\sigma(x) - x, y] = 0$. So, $[\sigma(x) - x, \mathfrak{g}_{-1}] = 0$. The transitivity(2.1) of \mathfrak{g} and the gradation of σ imply that $\sigma(x) - x \in \mathfrak{g}_{-k}$, $k \geq 1$. \square

Lemma 2.2. *Let $\mathfrak{g} = \oplus_{i \geq -h} \mathfrak{g}_i$ be a transitive and irreducible Lie superalgebra. If σ is a Hom-Lie superalgebra structure on \mathfrak{g} , and satisfies*

$$\sigma|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}} = \text{id}|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}},$$

then

$$\sigma = \text{id}|_{\mathfrak{g}}.$$

Proof. Let $i \geq 1$. Equation (2.2) and $\mathfrak{g}_{-i} = \mathfrak{g}_{-i}^i$ imply

$$\sigma|_{\mathfrak{g}_{-i}} = \text{id}|_{\mathfrak{g}_{-i}}. \quad (2.5)$$

Suppose $x \in \mathfrak{g}_i$, $y, z \in \mathfrak{g}_{\leq 0} = \oplus_{i=-h}^0 \mathfrak{g}_i$, by (2.4) and (2.5), one can deduce

$$[\sigma(x), [y, z]] = [x, [y, z]].$$

Since \mathfrak{g} is transitive and irreducible, then $\mathfrak{g}_{-1} = [\mathfrak{g}_0, \mathfrak{g}_{-1}]$. Together with Equation (2.4), it implies

$$[\sigma(x) - x, \mathfrak{g}_{\leq 0}] = 0.$$

Then $\sigma(x) - x = 0$, $\sigma|_{\mathfrak{g}_i} = \text{id}|_{\mathfrak{g}_i}$. Thus, we have $\sigma = \text{id}|_{\mathfrak{g}}$, that is, σ is trivial. \square

According to Lemma 2.2, to study the Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras, we can begin with their 0-th and -1-st components. In the remaining of this paper, we will directly use Equation (2.4) without further remarks.

3. Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras

3.1. $E(4,4)$

According to the studies reported in [20], there is a unique irreducible gradation over $E(4,4)$, such that $E(4,4)$ is the only inconsistent gradated algebra(its -1-st component is not purely odd) of the five exceptional simple Lie superalgebras. The 0-th component $E(4,4)_0$ is isomorphic to $\widehat{P}(4)$, which is the unique nontrivial center extension of $P(4)$. As $E(4,4)_0$ -module, $E(4,4)_{-1}$ is isomorphic to the nature module $\mathbb{F}^{4|4}$. In the followings, we will use the notations of $E(4,4)$, which are introduced in [21].

Let $C = (c_{ij}) \in \mathfrak{gl}_4(\mathbb{F})$ be a skew-symmetric matrix. $\tilde{C} = (\tilde{c}_{ij}) = (\varepsilon_{ijkl} c_{kl})$ stands for the Hodge dual of C , where ε_{ijkl} is the symbol of permutation $(1234) \mapsto (ijkl)$. For short, write $i' := i + 4$, $i = 1, 2, 3, 4$. Fix a basis of $\widehat{P}(4) : \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{I}$, where

$$\begin{aligned}\mathcal{A} &:= \{E_{ij} - E_{j'i'}, \quad 1 \leq i \neq j \leq 4\}, \\ \mathcal{B} &:= \{E_{ij'} + E_{ji'}, \quad 1 \leq i \leq j \leq 4\}, \\ \mathcal{C} &:= \{E_{i'j} - E_{j'i} - (\tilde{E}_{i'j} - \tilde{E}_{j'i}), \quad 1 \leq i < j \leq 4\}, \\ \mathcal{I} &:= \{I, E_{ii} - E_{i+1, i+1} - (E_{i'i'} - E_{i'+1, i'+1}), \quad 1 \leq i \leq 3\},\end{aligned}$$

and I is the unite matrix.

Suppose $E(4, 4)_{-1} = \langle v_i, v_{i'} | i = 1, \dots, 4 \rangle$, where $|v_i| = \bar{0}$, $|v_{i'}| = \bar{1}$.

Proposition 3.1. *If σ is a Hom-Lie superalgebra structure on $E(4, 4)$, then*

$$\sigma|_{E(4,4)_{-1}} = \text{id}|_{E(4,4)_{-1}}.$$

Proof. Let $i = 1, \dots, 8$, $k, j = 1, \dots, 4$ and $i \neq k'$, $j \neq i$. By using Equation (2.4), one can deduce

$$[\sigma(v_i), v_k] = [\sigma(v_i), [E_{kj} - E_{j'k'}, v_j]] = 0. \quad (3.1)$$

Similarly, one has

$$[\sigma(v_i), v_{k'}] = -[\sigma(v_i), [E_{jk} - E_{k'j'}, v_j]] = 0 \quad (3.2)$$

and

$$[\sigma(v_k), v_{k'}] = -[\sigma(v_k), [E_{k'j} - E_{j'k} - (\tilde{E}_{k'j} - \tilde{E}_{j'k}), v_j]] = -[\sigma(v_j), v_{j'}].$$

The arbitrariness of k, j in the last equation shows

$$[\sigma(v_i), v_{i'}] = 0, \quad i = 1, \dots, 4. \quad (3.3)$$

It follows from Equations (3.1)-(3.3) that

$$[\sigma(E(4, 4)_{-1}), E(4, 4)_{-1}] = 0.$$

Using the transitivity (2.1), one gets $\sigma(E(4, 4)_{-1}) \subseteq E(4, 4)_{-1}$. Recall that $|\sigma| = \bar{0}$, one may suppose

$$\sigma(v_i) = \sum_{m=1}^8 \lambda_m v_m, \lambda_m \in \mathbb{F}.$$

Now, for distinct $i, j, k, l = 1, 2, 3, 4$, suppose $(1234) \mapsto (ijkl)$ is an even permutation and

$$c = E_{k'j} - E_{j'k} - (E_{li'} - E_{il'}) \in \mathcal{C}, \quad a = E_{jl} - E_{l'j'} \in \mathcal{A}.$$

Obviously, $[a, v_i] = 0$, $[v_i, c] = 0$. Then

$$0 = [\sigma(v_i), [c, a]] = [\sigma(v_i), E_{kl} - E_{l'k} - (E_{ij'} - E_{ji'})] = -\lambda_l v_{k'} + \lambda_k v_{l'}.$$

Hence, $\lambda_k = \lambda_l = 0$, and then $\sigma(v_i) = \lambda_i v_i$, $i = 1, 2, 3, 4$.

The equation

$$0 = [\sigma(v_{i'}), [E_{kj} - E_{j'k'}, E_{jl'} + E_{lj'}]] = [\sigma(v_{i'}), E_{kl'} + E_{lk'}] = \lambda_{l'} v_k + \lambda_{k'} v_l,$$

implies $\lambda_{k'} = \lambda_{l'} = 0$. One has $\sigma(v_{i'}) = \lambda_{i'} v_{i'}$.

At last, let us prove $\sigma(v_i) = v_i$ and $\sigma(v_{i'}) = v_{i'}$. For distinct i, j, k , put

$$h = E_{jj} - E_{j+1,j+1} - (E_{kk} - E_{k+1,k+1}) \in \mathcal{I}, \quad a = E_{ji} - E_{j'i'} \in \mathcal{A}.$$

Clearly,

$$[\sigma(h), v_j] = [\sigma(h), [a, v_i]] = -[\sigma(v_i), a] = \lambda_i v_j. \quad (3.4)$$

Recall that σ is monomorphic, one may suppose σ^{-1} is a left linear inverse of σ , then

$$\sigma^{-1}[\sigma(h), v_j] = [h, \sigma^{-1}(v_j)] = \lambda_j^{-1} [h, v_j] = \lambda_j^{-1} v_j.$$

It implies that

$$[\sigma(h), v_j] = \sigma(\lambda_j^{-1} v_j) = v_j. \quad (3.5)$$

Comparing (3.4) with (3.5), one gets $\lambda_i = 1$. Analogously, one may check $\sigma(v_{i'}) = v_{i'}$.

Summing up the above, we have proved $\sigma|_{E(4,4)_{-1}} = \text{id}|_{E(4,4)_{-1}}$. \square

Proposition 3.2. *If σ is a Hom-Lie superalgebra structure on $E(4, 4)$, then*

$$\sigma|_{E(4,4)_0} = \text{id}|_{E(4,4)_0}.$$

Proof. Put $x \in E(4, 4)_0$, $v \in E(4, 4)_{-1}$. Using Proposition 3.1 and Lemma 2.1, one can deduce $\sigma(x) - x \in E(4, 4)_{-1}$. Note that \mathcal{A} and \mathcal{I} are generated (Lie product) by \mathcal{B} and \mathcal{C} , it is sufficient to prove the cases for $x \in \mathcal{B}$ and $x \in \mathcal{C}$.

Case 1 : Let $x = E_{ij'} + E_{ji'} \in \mathcal{B}$. Noting that $|\sigma| = \bar{0}$, one may suppose

$$\sigma(x) = x + \sum_{m=1}^4 \lambda_{m'} v_{m'}, \quad \lambda_{m'} \in \mathbb{F}.$$

Put

$$c = E_{k'l} - E_{l'k} - (\tilde{E}_{k'l} - \tilde{E}_{l'k}) \in \mathcal{C}, \quad b = E_{kk'} \in \mathcal{B},$$

where $k, l \neq i, j$. Then

$$0 = [\sigma(x), [b, c]] = \left[E_{ij'} + E_{ji'} + \sum_{m=1}^4 \lambda_{m'} v_{m'}, E_{kl} - E_{l'k'} \right] = \lambda_{k'} v_{l'}.$$

It implies that $\lambda_{k'} = 0$, $k \neq i, j$. Then,

$$\sigma(x) = x + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}. \quad (3.6)$$

Now, put

$$c = E_{i'k} - E_{k'i} - (\tilde{E}_{i'k} - \tilde{E}_{k'i'}) \in \mathcal{C}, \quad b = E_{ii'} \in \mathcal{B}, \quad k \neq i, j.$$

Using Equation (3.6), one may suppose $\sigma(b) = b + \mu_{i'} v_{i'}$, $\mu_{i'} \in \mathbb{F}$, then

$$0 = [\sigma(x), a] = [E_{ij'} + E_{ji'} + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}, E_{ik} - E_{k'i'}] = \lambda_{i'} v_{k'}.$$

So $\lambda_{i'} = 0$. That is, $\sigma(x) = x + \lambda_{j'} v_{j'}$. Similarly, put

$$c = E_{j'k} - E_{k'j} - (\tilde{E}_{j'k} - \tilde{E}_{k'j'}) \in \mathcal{C}, \quad b = E_{jj'} \in \mathcal{B}.$$

One may obtain $\lambda_{j'} = 0$. Thus, we proved $\sigma(x) = x$ for any $x \in \mathcal{B}$.

Case 2 : Let $x = E_{i'l} - E_{j'i} - (E_{kl'} - E_{lk'}) \in \mathcal{C}$, where $\varepsilon_{ijkl} = 1$. Noting that $|\sigma| = \bar{0}$, one may assume $\sigma(x) = x + \sum_{m=1}^4 \mu_{m'} v_{m'}$, $\mu_{m'} \in \mathbb{F}$.

Firstly, put

$$c = E_{k'l} - E_{l'k} - (E_{ji'} - E_{ij'}) \in \mathcal{C}, \quad b = E_{kk'} \in \mathcal{B},$$

where $\varepsilon_{ijkl} = 1$. Equation (2.4) and the result $\sigma(b) = b$ obtained in Case 1 imply

$$0 = [\sigma(x), [c, b]] = [x + \sum_{m=1}^4 \mu_{m'} v_{m'}, E_{kl} - E_{l'k'}] = c_0 + \mu_{k'} v_{l'}, \quad c_0 \in \mathcal{C}.$$

The equation above shows $\mu_{k'} = 0$. By the arbitrariness of $k \neq i, j$, one has

$$\sigma(x) = x + \mu_{i'} v_{i'} + \mu_{j'} v_{j'}. \quad (3.7)$$

Secondly, put

$$c = E_{l'j} - E_{j'l} - (E_{ik'} - E_{k'i}) \in \mathcal{C}, \quad b = E_{jj'}.$$

Using Equation (3.7), one may suppose

$$\sigma(c) = c + \gamma_{l'} v_{l'} + \gamma_{j'} v_{j'}, \quad \gamma_{l'}, \gamma_{j'} \in \mathbb{F}.$$

On one hand,

$$[\sigma(x), [b, c]] = -[\sigma(b), [c, x]] - [\sigma(c), [x, b]] = E_{l'i} - E_{i'l} - (E_{kj'} - E_{jk'}) - \gamma_{j'} v_{i'}.$$

On the other hand,

$$\begin{aligned} [\sigma(x), [b, c]] &= [E_{i'j} - E_{j'i} - (E_{kl'} - E_{lk'}) + \mu_{i'} v_{i'} + \mu_{j'} v_{j'}, E_{l'j} - E_{j'l}] \\ &= E_{l'i} - E_{i'l} - (E_{kj'} - E_{jk'}) - \mu_{j'} v_{l'}. \end{aligned}$$

Since $i \neq l$, one has $\gamma_{j'} = \mu_{j'} = 0$. Then $\sigma(x) = x + \lambda_{i'} v_{i'}$.

At last, put

$$c = E_{i'k} - E_{k'i} - (E_{lj'} - E_{jl'}) \in \mathcal{C}, \quad b = E_{ii'} \in \mathcal{B}.$$

We can obtain $\lambda_{i'} = 0$ in the same way. Thus, $\sigma(x) = x$ is proved.

Summing up, we have proved $\sigma(x) = x$ for any $x \in E(4, 4)_0$, that is, $\sigma|_{E(4, 4)_0} = \text{id}|_{E(4, 4)_0}$. \square

3.2. E(3,6), E(5,10) and E(3,8)

Before studying the Hom-Lie superalgebra structures on $E(3, 6)$, $E(5, 10)$ and $E(3, 8)$, we would like to review their algebraic structures briefly [18, 20, 22, 23]. As we know in [18], the even part $E(5, 10)_0$ of $E(5, 10)$ is isomorphic to the Lie algebra S_5 , which is of divergence 0 polynomial vector fields on \mathbb{F}^5 , i.e., polynomial vector fields annihilating the volume form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_5$. As S_5 -module, the odd part $E(5, 10)_1$ of $E(5, 10)$ is isomorphic to $d\Omega^1(5)$, the space of closed polynomial differential 2-forms on \mathbb{F}^5 . Next, we keep the notations in [22]:

$$d_{ij} := dx_i \wedge dx_j, \quad \partial_i := \partial/\partial x_i.$$

One may denote $D \in E(5, 10)_{\bar{0}}$ through

$$D = \sum_{i=1}^5 f_i \partial_i, \quad \text{where } f_i \in \mathbb{F}[[x_1, x_2, \dots, x_5]], \quad \sum_{i=1}^5 \partial_i(f_i) = 0,$$

and denote $E \in E(5, 10)_{\bar{1}}$ through

$$E = \sum_{i,j=1}^5 f_{ij} d_{ij}, \quad \text{where } f_{ij} \in \mathbb{F}[[x_1, x_2, \dots, x_5]], \quad dE = 0. \quad (3.8)$$

The bracket in $E(5, 10)_{\bar{1}}$ is defined by

$$[f d_{ij}, g d_{kl}] = \varepsilon_{tijk} f g \partial_l,$$

where ε_{tijk} is the sign of permutation $(tijk)$ when $\{tijk\} = \{12345\}$ and zero otherwise. The bracket of $E(5, 10)_{\bar{0}}$ with $E(5, 10)_{\bar{1}}$ is defined by the usual action of vector fields on differential forms. The irreducible consistent \mathbb{Z} -gradation over $E(5, 10)$ is defined by letting (see [22])

$$\text{zd}(x_i) = 2, \quad \text{zd}(d) = -\frac{5}{2}, \quad \text{zd}(dx_i) = -\frac{1}{2}.$$

Then $E(5, 10) = \oplus_{i \geq -2} \mathfrak{g}_i$, where

$$\begin{aligned} \mathfrak{g}_0 &\simeq \mathfrak{sl}(5) = \langle x_i \partial_j, x_k \partial_l - x_{k+1} \partial_{l+1} \mid i, j = 1, \dots, 5, i \neq j, k = 1, \dots, 4 \rangle, \\ \mathfrak{g}_{-1} &\simeq \wedge^2 \mathbb{F}^5 = \langle d_{ij} \mid 1 \leq i < j \leq 5 \rangle, \\ \mathfrak{g}_{-2} &\simeq \mathbb{F}^{5*} = \langle \partial_i \mid i = 1, \dots, 5 \rangle. \end{aligned}$$

The exceptional simple Lie superalgebra $E(3, 6)$ is a subalgebra of $E(5, 10)$. The irreducible consistent \mathbb{Z} -gradation over $E(3, 6)$ is induced by the \mathbb{Z} -gradation of $E(5, 10)$ above. Let $h_1 = x_1 \partial_1 - x_2 \partial_2$, $h_2 = x_2 \partial_2 - x_3 \partial_3$, $h_3 = x_4 \partial_4 - x_5 \partial_5$, $h_4 = -x_2 \partial_2 - x_3 \partial_3 + 2x_5 \partial_5$. Then $E(3, 6) = \oplus_{i \geq -2} \mathfrak{g}_i$, where

$$\begin{aligned} \mathfrak{g}_0 &\simeq \mathfrak{gl}(3) \oplus \mathfrak{sl}(2) \\ &= \langle x_i \partial_j, x_k \partial_l, h_m \mid i, j = 1, 2, 3, i \neq j, k, l = 4, 5, k \neq l, m = 1, \dots, 4 \rangle, \\ \mathfrak{g}_{-1} &\simeq \mathbb{F}^3 \otimes \mathbb{F}^2 = \langle d_{ij} \mid i = 1, 2, 3, j = 4, 5 \rangle, \\ \mathfrak{g}_{-2} &\simeq \mathbb{F}^{3*} = \langle \partial_i \mid i = 1, 2, 3 \rangle. \end{aligned}$$

The exceptional simple Lie superalgebra $E(3, 8)$, which is strikingly similar to $E(3, 6)$, carries a unique irreducible consistent \mathbb{Z} -gradation[23] defined by

$$\text{zd}(x_i) = -\text{zd}(\partial_i) = \text{zd}(dx_i) = 2, \quad i = 1, 2, 3; \quad \text{zd}(x_4) = \text{zd}(x_5) = -3.$$

Then $E(3, 8) = \oplus_{i \geq -3} \mathfrak{g}_i$, where

$$\mathfrak{g}_j = E(3, 6)_j, \quad j = 0, -1, -2; \quad \mathfrak{g}_{-3} \simeq \mathbb{F}^2 \simeq \langle dx_4, dx_5 \rangle. \quad (3.9)$$

Hereafter, \mathfrak{g} denotes $E(5, 10)$, $E(3, 6)$ or $E(3, 8)$ unless otherwise noted. We establish a technical lemma, which can be verified directly.

Lemma 3.3. *If σ is a Hom-Lie superalgebra structure on \mathfrak{g} , then*

- (1) $[\sigma(d_{ij}), d_{il}] = \begin{cases} 0, & \text{if } d_{ij}, d_{il} \in E(5, 10), \\ -[\sigma(d_{il}), d_{ij}], & \text{if } d_{ij}, d_{il} \in E(3, 6) \text{ or } E(3, 8); \end{cases}$
- (2) $[\sigma(d_{ij}), d_{kj}] = 0$, in particular, $[\sigma(d_{ij}), d_{ij}] = 0$;
- (3) $[\sigma(d_{ij}), d_{kl}] = [d_{ij}, \sigma(d_{kl})]$ for distinct i, j, k, l .

Proposition 3.4. Let \mathfrak{g} be Lie superalgebra $E(5, 10)$, $E(3, 6)$ or $E(3, 8)$. If σ is a Hom-Lie superalgebra structure on \mathfrak{g} , then

$$\sigma|_{\mathfrak{g}_{-1}} = \text{id}|_{\mathfrak{g}_{-1}}.$$

Proof. First, let us prove $\sigma|_{\mathfrak{g}_{-1}} = \lambda \text{id}|_{\mathfrak{g}_{-1}}$, where $\lambda \in \mathbb{F}$.

Case 1 : $\mathfrak{g} = E(5, 10)$. Noting that the gradation over $E(5, 10)$ is consistent and $|\sigma| = \bar{0}$, $|d_{ij}| = \bar{1}$, one may suppose

$$\sigma(d_{ij}) = \sum_{1 \leq m < n \leq 5} f_{mn} d_{mn}, \quad \text{where } f_{mn} \in \mathbb{F}[[x_1, \dots, x_5]].$$

By Lemma 3.3 (1), one has

$$\begin{aligned} [\sigma(d_{ij}), d_{il}] &= \left[\sum_{1 \leq m < n \leq 5} f_{mn} d_{mn}, d_{il} \right] = \left[\sum_{m, n \neq i, l} f_{mn} d_{mn}, d_{il} \right] \\ &= \sum_{m, n \neq i, l} [f_{mn} d_{mn}, d_{il}] = \sum_{m, n \neq i, l} \varepsilon_{qmnil} f_{mn} \partial_q = 0. \end{aligned}$$

The arbitrariness of l shows

$$f_{mn} = 0, \quad \text{where } m, n \neq i.$$

Similarly, by Lemma 3.3 (2), one gets

$$f_{mn} = 0, \quad \text{where } m, n \neq j.$$

Thus,

$$\sigma(d_{ij}) = f_{ij} d_{ij}.$$

In view of $\sigma(d_{ij}) \in E(5, 10)_{\bar{1}} \simeq d\Omega^1(5)$ and (3.8), one has

$$0 = d(\sigma(d_{ij})) = d(f_{ij} d_{ij}) = \sum_{m=1}^5 \partial_m(f_{ij}) dx_m \wedge dx_i \wedge dx_j.$$

Therefore, $\partial_m(f_{ij}) = 0$ for $m \neq i, j$, that is $f_{ij} \in \mathbb{F}[[x_i, x_j]]$. So one may suppose $\sigma(d_{kl}) = f_{kl} d_{kl}$, where $f_{kl} \in \mathbb{F}[[x_k, x_l]]$. Then

$$[\sigma(d_{ij}), d_{kl}] = [f_{ij} d_{ij}, d_{kl}] = \varepsilon_{qijkl} f_{ij} \partial_q,$$

$$[\sigma(d_{kl}), d_{ij}] = [f_{kl} d_{kl}, d_{ij}] = \varepsilon_{qkl ij} f_{kl} \partial_q.$$

For distinct i, j, k, l , Lemma 3.3 (3) implies $f_{ij} = f_{kl} \in \mathbb{F}$. Noting that there exists t such that $t \neq i, j, k, l$, one may obtain $f_{kt} = f_{ij}$ in the same way. Then $f_{ij} = \lambda \in \mathbb{F}$ for any $i, j = 1, 2, 3, 4, 5$ and $i \neq j$. Thus, we proved $\sigma(d_{ij}) = \lambda d_{ij}$.

Case 2 : $\mathfrak{g} = E(3, 6)$ or $E(3, 8)$. Noting that the gradation over \mathfrak{g} is consistent and $|\sigma| = \bar{0}$, $|d_{ij}| = \bar{1}$, one may suppose

$$\sigma(d_{ij}) = \sum_{m,n} f_{mn} d_{mn}, \quad \text{where } m = 1, 2, 3, n = 4, 5, f_{mn} \in \mathbb{F}[[x_1, \dots, x_5]].$$

By Lemma 3.3 (2), one can deduce

$$[\sigma(d_{ij}), d_{kj}] = \left[\sum_{m,n} f_{mn} d_{mn}, d_{kj} \right] = \left[\sum_{m \neq k, n \neq j} f_{mn} d_{mn}, d_{kj} \right] = 0.$$

It implies $f_{mt} = 0$, $m \neq k$, $t \neq j$. By the arbitrariness of k , one may suppose

$$\sigma(d_{ij}) = \sum_{m=1}^3 f_{mj} d_{mj}, \quad \sigma(d_{il}) = \sum_{m=1}^3 f_{ml} d_{ml}, \quad f_{mj}, f_{ml} \in \mathbb{F}[[x_1, \dots, x_5]].$$

Noting that $[\sigma(d_{il}), d_{kj}] = [\sigma(d_{il}), [x_k \partial_i, d_{ij}]] = -[\sigma(d_{ij}), d_{kl}]$, the simple computations show $f_{mj} = f_{ml}$ for $m = 1, 2, 3$. In view of $\sigma(d_{ij}) \in \mathfrak{g}_{\bar{1}} \subseteq d\Omega^1(5)$ and (3.8), one could further deduce

$$f_{mj} = f_{ml} \in \mathbb{F}[[x_1, x_2, x_3]]. \quad (3.10)$$

For distinct $i, k, q = 1, 2, 3$,

$$\begin{aligned} 0 &= [\sigma(d_{ij}), [x_i \partial_q, x_q \partial_k]] = [\sigma(d_{ij}), x_i \partial_k] = \left[\sum_{m=1}^3 f_{mj} d_{mj}, x_i \partial_k \right] \\ &= - \sum_{m=1}^3 (x_i \partial_k (f_{mj}) d_{mj} + f_{mj} x_i \partial_k (d_{mj})) \\ &= -(x_i \partial_k (f_{ij}) + f_{kj}) d_{ij} - x_i \partial_k (f_{kj}) d_{kj} - x_i \partial_k (f_{qj}) d_{qj}. \end{aligned}$$

Therefore,

$$f_{kj} = -x_i \partial_k (f_{ij}). \quad (3.11)$$

$$\partial_k (f_{kj}) = \partial_k (f_{qj}) = 0, \quad (3.12)$$

From (3.10)-(3.12) and the arbitrariness of k , one has $f_{mj} = 0$, $m \neq i$. Thus, one may suppose

$$\sigma(d_{ij}) = f_i d_{ij}, \quad \sigma(d_{kl}) = f_k d_{kl}, \quad \text{where } f_i \in \mathbb{F}[[x_i]], f_k \in \mathbb{F}[[x_k]].$$

Then, for distinct i, j, k, l, q ,

$$[\sigma(d_{ij}), d_{kl}] = [f_i d_{ij}, d_{kl}] = \varepsilon_{qijkl} f_i \partial_q,$$

$$[d_{ij}, \sigma(d_{kl})] = [d_{ij}, f_k d_{kl}] = \varepsilon_{qijkl} f_k \partial_q.$$

By Lemma 3.3 (3), $f_i = f_k \in \mathbb{F}$. Thus, we have proved $\sigma(d_{ij}) = \lambda d_{ij}$ for some $\lambda \in \mathbb{F}$.

Now, let us prove $\lambda = 1$ for $\sigma|_{\mathfrak{g}_{-1}} = \lambda \text{id}|_{\mathfrak{g}_{-1}}$. Suppose $x = x_i \partial_l$, $y = x_l \partial_q$, $z = d_{qj}$. Then

$$[\sigma(x), d_{lj}] = [\sigma(x), [y, z]] = -[\sigma(y), [z, x]] - [\sigma(z), [x, y]] = \lambda d_{ij}.$$

Noting that σ is a monomorphism, one may suppose σ^{-1} is a left linear inverse of σ . Then the equation above implies

$$\sigma^{-1}[\sigma(x), d_{lj}] = [x, \lambda^{-1}d_{lj}] = \lambda^{-1}d_{ij} = d_{ij}.$$

Thus, $\lambda = 1$, that is, $\sigma(d_{ij}) = d_{ij}$. The proof is completed. \square

Proposition 3.5. *Let \mathfrak{g} be Lie superalgebra $E(5, 10)$, $E(3, 6)$ or $E(3, 8)$. If σ is a Hom-Lie superalgebra structure on \mathfrak{g} , then*

$$\sigma|_{\mathfrak{g}_0} = \text{id}|_{\mathfrak{g}_0}.$$

Proof. Case 1 : $\mathfrak{g} = E(5, 10)$. Note that $E(5, 10)_0$ is generated (Lie product) by $\{x_i\partial_j | i, j = 1, \dots, 5, i \neq j\}$, it is sufficient to prove $\sigma(x) = x$ only for $x = x_i\partial_j$. By Proposition 3.4, Lemma 2.1 and $|\sigma| = \bar{0}$, we have $\sigma(x_i\partial_j) - x_i\partial_j \in \mathfrak{g}_{-2}$. Then one may suppose

$$\sigma(x_i\partial_j) = x_i\partial_j + \sum_{m=1}^5 \lambda_m \partial_m, \quad \lambda_m \in \mathbb{F}.$$

For distinct i, j, q, k, l ,

$$0 = [\sigma(x_i\partial_j), [x_q\partial_k, x_k\partial_l]] = \left[x + \sum_{m=1}^5 \lambda_m \partial_m, x_q\partial_l \right] = \lambda_q \partial_l.$$

Hence, $\lambda_q = 0$. The arbitrariness of $q \neq i, j$ shows

$$\sigma(x_i\partial_j) = x_i\partial_j + \lambda_i \partial_i + \lambda_j \partial_j.$$

Similarly,

$$0 = [\sigma(x_i\partial_j), [x_i\partial_l, x_l\partial_k]] = [x + \lambda_i \partial_i + \lambda_j \partial_j, x_i\partial_k] = \lambda_i \partial_k$$

implies $\lambda_i = 0$. Therefore,

$$\sigma(x_i\partial_j) = x_i\partial_j + \lambda_j \partial_j. \tag{3.13}$$

On one hand,

$$[\sigma(x_i\partial_j), [x_j\partial_i, x_i\partial_k]] = [x_i\partial_j + \lambda_j \partial_j, x_j\partial_k] = x_i\partial_k + \lambda_j \partial_k. \tag{3.14}$$

On the other hand, using (2.4) and (3.13), one can derive

$$[\sigma(x_i\partial_j), [x_j\partial_i, x_i\partial_k]] = -[\sigma(x_i\partial_k), x_i\partial_i - x_j\partial_j] = x_i\partial_k. \tag{3.15}$$

Comparing (3.14) with (3.15), one gets $\lambda_j = 0$. By (3.13), we have proved $\sigma(x_i\partial_j) = x_i\partial_j$.

Case 2 : $\mathfrak{g} = E(3, 6)$ or $E(3, 8)$. Firstly, let us show $\sigma(x_i\partial_j) = x_i\partial_j$ for any $x_i\partial_j \in \mathfrak{g}_0$. Similar to the above, one may suppose

$$\sigma(x_i\partial_j) = x_i\partial_j + \sum_{m=1}^3 \lambda_m \partial_m, \quad \lambda_m \in \mathbb{F}.$$

Case 2.1 : Let $i, j = 1, 2, 3$. For distinct i, j, k , by (2.4),

$$0 = [\sigma(x_i\partial_j), [x_i\partial_k, x_k\partial_j]] = [\sigma(x_i\partial_j), x_i\partial_j] = \left[x_i\partial_j + \sum_{m=1}^3 \lambda_m \partial_m, x_i\partial_j \right] = \lambda_i \partial_j.$$

Consequently, $\lambda_i = 0$. One may suppose $\sigma(x_k \partial_j) = x_k \partial_j + \sum_{m \neq k}^3 \mu_m \partial_m$, On one hand,

$$[\sigma(x_i \partial_j), x_k \partial_i] = \left[x_i \partial_j + \sum_{m \neq i} \lambda_m \partial_m, x_k \partial_i \right] = -x_k \partial_j + \lambda_k \partial_i.$$

On the other hand, by (2.4), one deduce

$$\begin{aligned} [\sigma(x_i \partial_j), x_k \partial_i] &= [\sigma(x_i \partial_j), [x_k \partial_j, x_j \partial_i]] \\ &= -[\sigma(x_k \partial_j), [x_j \partial_i, x_i \partial_j]] - [\sigma(x_j \partial_i), [x_i \partial_j, x_k \partial_j]] \\ &= -x_k \partial_j - \mu_j \partial_j + \mu_i \partial_i. \end{aligned}$$

Comparing the two equations above, one gets $\mu_j = 0$ and $\lambda_k = \mu_i$. In view of the arbitrariness of i, j, k , one may assume

$$\sigma(x_i \partial_j) = x_i \partial_j + \lambda \partial_k, \quad \sigma(x_k \partial_j) = x_k \partial_j + \lambda \partial_i, \quad i, j, k \text{ are distinct.}$$

Then

$$\sigma(x_i \partial_j) = \sigma[x_i \partial_k, x_k \partial_j] = [\sigma(x_i \partial_k), \sigma(x_k \partial_j)] = x_i \partial_j - \lambda \partial_k.$$

The two equations above imply $\lambda = 0$, and then $\sigma(x_i \partial_j) = x_i \partial_j$, $i, j = 1, 2, 3$.

Case 2.2: Let $i, j = 4, 5$. For distinct $k, l, q = 1, 2, 3$,

$$0 = [\sigma(x_i \partial_j), [x_k \partial_q, x_q \partial_l]] = [\sigma(x_i \partial_j), x_k \partial_l] = \left[x_i \partial_j + \sum_{m=1}^3 \lambda_m \partial_m, x_k \partial_l \right] = \lambda_k \partial_l.$$

Therefor, $\lambda_k = 0$. The arbitrariness of k implies $\sigma(x_i \partial_j) = x_i \partial_j$.

Next, we will prove $\sigma(h_m) = h_m$ for $m = 1, 2, 3, 4$. Similarly, suppose $\sigma(h_m) = h_m + \sum_{k=1}^3 \lambda_k \partial_k$. Noting that h_m is an element of the basis of \mathfrak{g}_0 's Cartan subalgebra, one may assume $[h_m, x_i \partial_j] = \gamma x_i \partial_j$, $\gamma \in \mathbb{F}$, $i, j = 1, 2, 3$. By the result obtained in Case 2.1, one can deduce

$$\sigma[h_m, x_i \partial_j] = \gamma \sigma(x_i \partial_j) = \gamma x_i \partial_j.$$

On the other hand,

$$\sigma[h_m, x_i \partial_j] = [\sigma(h_m), \sigma(x_i \partial_j)] = \left[h_m + \sum_{k=1}^3 \lambda_k \partial_k, x_i \partial_j \right] = \gamma x_i \partial_j + \lambda_i \partial_j.$$

The two equations above imply that $\lambda_i = 0$ for $i = 1, 2, 3$. Thus, we proved $\sigma(h_m) = h_m$ for $m = 1, 2, 3, 4$.

Summing up, we have proved $\sigma(x) = x$ for any $x \in \mathfrak{g}_0$, that is, $\sigma|_{\mathfrak{g}_0} = \text{id}|_{\mathfrak{g}_0}$. \square

3.3. $E(1,6)$

The exceptional simple Lie superalgebra $E(1,6)$ is a subalgebra of $K(1,6)$. The principal gradation over $K(1,6)$ (see [20]) induces an irreducible consistent \mathbb{Z} -gradation over $E(1,6)$.

Moreover, $E(1, 6)$ has the same non-positive \mathbb{Z} -graded components as $K(1, 6)$: $E(1, 6)_j = K(1, 6)_j$, $j \leq 0$. Let x be even and ξ_i , $i = 1, \dots, 6$, be odd indeterminates. Then

$$\begin{aligned} E(1, 6)_0 &\simeq \mathbf{cspo}(6) = \langle x, \xi_i \xi_j \mid i, j = 1, \dots, 6, i \neq j \rangle, \\ E(1, 6)_{-1} &\simeq \mathbb{F}^6 = \langle \xi_i \mid i = 1, \dots, 6 \rangle, \\ E(1, 6)_{-2} &\simeq \mathbb{F}. \end{aligned}$$

For $f, g \in E(1, 6)$, the bracket product is defined as

$$\begin{aligned} [f, g] &= \left(2f - \sum_{i=1}^6 \xi_i \partial_i(f) \right) \partial_x(g) - (-1)^{|f||g|} \left(2g - \sum_{i=1}^6 \xi_i \partial_i(g) \right) \partial_x(f) \\ &\quad + (-1)^{|f|} \sum_{i=1}^6 \partial_i(f) \partial_i(g). \end{aligned}$$

Proposition 3.6. *If σ is a Hom-Lie superalgebra structure on $E(1, 6)$, then*

$$\sigma|_{E(1,6)_{-1}} = \text{id}|_{E(1,6)_{-1}}.$$

Proof. For $i, j, k = 1, \dots, 6$ and $j \neq i, k$, by (2.4),

$$[\sigma(\xi_i), \xi_k] = [\sigma(\xi_i), [\xi_j \xi_k, \xi_j]] = \delta_{k,i} [\sigma(\xi_j), \xi_j].$$

It implies that $[\sigma(\xi_i), \xi_k] = 0$ for any $k, i = 1, \dots, 6$. From $|\sigma| = \bar{0}$ and the transitivity of $E(1, 6)$, one can deduce $\sigma(E(1, 6)_{-1}) \subseteq E(1, 6)_{-1}$. So one may assume $\sigma(\xi_i) = \sum_{k=1}^6 \lambda_k \xi_k$, $\lambda_k \in \mathbb{F}$. By using Equation (2.4), for distinct $i, j, s, t = 1, \dots, 6$, one can deduce

$$0 = [\sigma(\xi_i), [\xi_j \xi_t, \xi_s \xi_t]] = [\sigma(\xi_i), \xi_j \xi_s] = \lambda_j \xi_s - \lambda_s \xi_j.$$

Then $\lambda_k = 0$ for any $k \neq i$. That is $\sigma(\xi_i) = \lambda_i \xi_i$. By using Equation (2.4) again, one can deduce

$$-\lambda_j = [\sigma(\xi_i), [\xi_j \xi_i, \xi_j]] = [\sigma(\xi_i), \xi_i] = [\lambda_i \xi_i, \xi_i] = -\lambda_i.$$

Moreover, one knows $\sigma(\xi_i) = \lambda \xi_i$ for any $i = 1, \dots, 6$. The remaining work is to prove $\lambda = 1$. For distinct i, j, t ,

$$[\sigma(\xi_t \xi_j), \xi_t] = [\sigma(\xi_t \xi_j), [\xi_i \xi_t, \xi_i]] = -[\sigma(\xi_i), [\xi_t \xi_j, \xi_i \xi_t]] = [\lambda \xi_i, \xi_j \xi_i] = \lambda \xi_j. \quad (3.16)$$

Suppose σ^{-1} is a left inverse of the monomorphism σ , then

$$\sigma^{-1}([\sigma(\xi_t \xi_j), \xi_t]) = [\xi_t \xi_j, \sigma^{-1}(\xi_t)] = \lambda^{-1} \xi_j.$$

Hence

$$[\sigma(\xi_t \xi_j), \xi_t] = \sigma(\lambda^{-1} \xi_j) = \xi_j. \quad (3.17)$$

Equations (3.16) and (3.17) imply $\lambda = 1$. The proof is completed. \square

Proposition 3.7. *If σ is a Hom-Lie superalgebra structure on $E(1, 6)$, then*

$$\sigma|_{E(1,6)_0} = \text{id}|_{E(1,6)_0}.$$

Proof. For any $y \in E(1, 6)_0$, by Proposition 3.6 and Lemma 2.1, $\sigma(y) - y \in E(1, 6)_{-2} \simeq \mathbb{F}$. So, one may suppose $\sigma(y) = y + \lambda$, $\lambda \in \mathbb{F}$. Then

$$[\sigma(y), x] = [y + \lambda, x] = [\lambda, x] = 2\lambda. \quad (3.18)$$

If $y = x$, noting that x is an element of a basis of $E(1, 6)_0$'s Cartan subalgebra, one may assume

$$[\sigma(x), x] = \mu\sigma(x) = \mu(x + \lambda), \quad \mu \in \mathbb{F}. \quad (3.19)$$

Comparing (3.18) and (3.19), one has $\mu = 0$, and then $\lambda = 0$, that is $\sigma(x) = x$.

If $y = \xi_i \xi_j$,

$$[\sigma(\xi_i \xi_j), x] = [\sigma(\xi_i \xi_j), \sigma(x)] = \sigma[\xi_i \xi_j, x] = 0. \quad (3.20)$$

From (3.18) and (3.20), one gets $\lambda = 0$ again. The proof is completed. \square

3.4. Result

Propositions 3.1, 3.2, 3.4-3.7 show that the Hom-Lie superalgebra structures on the 0-th and -1 -st \mathbb{Z} -components of each infinite-dimensional exceptional simple Lie superalgebras are trivial. Together with Lemma 2.2, it is sufficient to draw the following conclusion.

Theorem 3.8. *There is only the trivial Hom-Lie superalgebra structure on each exceptional simple Lie superalgebras.*

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